LOG-CANONICAL FORMS AND LOG CANONICAL SINGULARITIES

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Dedicated to H. Grauert on occasion of his 70's birthday

Abstract. For a normal subvariety V of $\mathbb{C}^n$ with a good $\mathbb{C}^*$-action we give a simple characterization for when it has only log canonical, log terminal or rational singularities. Moreover we are able to give formulas for the plurigenera of isolated singular points of such varieties and of the logarithmic Kodaira dimension of $V\backslash\{0\}$. For this purpose we introduce sheaves of m-canonical and $L^{2,m}$-canonical forms on normal complex spaces. For the case of affine varieties with good $\mathbb{C}^*$-action we give an explicit formula for these sheaves in terms of the grading of the dualizing sheaf and its tensor powers.

Contents

Introduction

Let X be a normal complex space and D a reduced Weil divisor on X. In this paper we will associate to the pair (X, D) two sheaves $\mathcal{L}^m_{X,D}$ and $\mathcal{L}^{2,m}_{X,D}$, which we

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call the sheaves of logarithmic m-, resp. L^2-m-canonical forms. The construction is in brief as follows: let $\pi : X' \to X$ be a resolution of singularities such that $D' := \pi^{-1}(D \cup \operatorname{Sing} X)_{\mathrm{red}}$ is an SNC (=simple normal crossing) divisor. The sheaves

$$\mathcal{L}^m_{X,D} := \pi_*(\mathcal{O}_{X'}(m(K_{X'} + D'))) \quad \text{and} \quad \mathcal{L}^{2,m}_{X,D} := \pi_*(\mathcal{O}_{X'}(mK_{X'} + (m-1)D'))$$

are then independent of the choice of resolution. For instance, the L^2-m-canonical forms are just those m-canonical forms on $X\backslash(D \cup \operatorname{Sing} X)$ that are locally L^2 at the points of $D \cup \operatorname{Sing} X$, see [17]. The sheaf $\mathcal{L}^{2,1}_X := \mathcal{L}^{2,1}_{X,0}$ was previously studied in the paper of Grauert-Riemenschneider [7] (it is called there the canonical sheaf of X).

A motivation to study these sheaves is that they allow simple characterizations for when a singularity is rational, log terminal or log canonical. As an example, by a result of Kempf [11] a normal complex algebraic variety X has rational singularities if and only if it is Cohen-Macaulay and satisfies $\mathcal{L}^{2,1}_X \cong \mathcal{O}_X(K_X)$. In analogy with this result we will show: if (X, x) is a normal complex singularity and D is a reduced Weil divisor with $K_X + D$ being $\mathbb{Q}$-Cartier, then (X, D) has a log canonical singularity at x if and only if the stalks $(\mathcal{L}^m_{X,D})_x$ and $\mathcal{O}_X(m(K_X + D))_x$ are equal for all $m \geq 1$. In the case when $D = 0$ a similar characterization holds for log terminal singularities, see 1.17.

The main application of these log canonical sheaves is to the case of affine varieties $V = \operatorname{Spec} A$ for which the coordinate ring $A = \bigoplus_{i \geq 0} A_i$ is a non-negatively graded $\mathbb{C}$-algebra. Let $D \subseteq V$ be a reduced Weil divisor and assume for simplicity that D is given by an equation $P = 0$, where P is homogeneous of degree d. The modules of sections

$$L^m_{A,D} := H^0(V, \mathcal{L}^m_{V,D}) \quad \text{and} \quad L^{2,m}_{A,D} := H^0(V, \mathcal{L}^{2,m}_{V,D})$$

as well as the reflexive hull $\omega_A^{[m]}$ of the m-th tensor power of the dualizing module ω_A then carry natural gradings. We will show: *if $V^* := V\backslash V(A_+)$ is smooth and $D \cap V^*$ is an SNC divisor then*

$$L^m_{A,D} = (\omega_A^{[m]})_{\geq md} \quad \text{and} \quad L^{2,m}_{A,D} = (\omega_A^{[m]})_{>(m-1)d}\,;$$

see 2.15 for a more general statement. This implies for instance that (V, D) has log canonical singularities if and only if $\omega_A^{[m]}$ has no elements of degree $< md$. Similar characterizations hold for the properties log terminal and rational, see 2.16. For the latter case this is a result of [4] and [19].

As another application we obtain formulas for the plurigenera of the singularities of V, where V is as above. For instance, the plurigenus δ_m introduced in [20] is given by

$$\delta_m(X, p) = \dim_{\mathbb{C}} (\omega_A^{[m]})_{\leq 0},$$

see 2.22. Applying this to complete intersections we recover a result of Morales [15]. A final application concerns the logarithmic plurigenera and the Kodaira dimension of $V\backslash D$. With the assumptions on (V, D) as above, assume moreover that $A_0 \cong \mathbb{C}$ so that the $\mathbb{C}^*$-action corresponding to the grading is good. Then the logarithmic plurigenus $\bar{p}_m(V\backslash D)$ is given by the dimension of $\omega^{[m]}_{md}$, see 2.26.

The paper is organized as follows. In Section 1 we introduce the sheaves of log canonical forms and study their basic properties. In particular we show the characterization of log canonical and log terminal singularities in terms of log canonical forms mentioned above.

Sections 2.2–2.5 contain the applications to affine varieties with $\mathbb{C}^*$-action as described above In Section 2.1 we provide some material concerning equivariant completions and weighted blowups of quasihomogeneous affine varieties.

In this paper we work in the category of complex spaces and varieties over $\mathbb{C}$. However, the principal results remain valid for algebraic varieties over any field of characteristic zero.

1. Logarithmic m-canonical forms on singular spaces

1.1. Logarithmic m-canonical forms on smooth varieties.

Notation 1.1. Let X be a complex manifold and $D \subseteq X$ a divisor with simple normal crossings (SNC in brief). Consider the sheaf of logarithmic m-canonical forms

$$\mathcal{L}^m_{X,D} := \mathcal{O}_X(mK_X + mD)$$

and the sheaf of logarithmic L^2-m-canonical forms

$$\mathcal{L}^{2,m}_{X,D} := \mathcal{O}_X(mK_X + (m-1)D) = \mathcal{L}^m_{X,D}(-D)\,.$$

If $x_1, \ldots, x_n$ are local coordinates around a point, say, $p \in X$ with $D = \{x_1 \cdot \ldots \cdot x_k = 0\}$, then near p the $\mathcal{O}_X$-module $\mathcal{L}^m_{X,D}$ is generated by

$$\omega_m = \left(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_k}{x_k} \wedge dx_{k+1} \wedge \cdots \wedge dx_n\right)^{\otimes m},$$

whereas the sheaf $\mathcal{L}^{2,m}_{X,D}$ is generated by $x_1 \cdot \ldots \cdot x_k \cdot \omega_m$. As explained in [17, Thm. 2.1], the forms in $H^0(X, \mathcal{L}^{2,m}_{X,D})$ are just the meromorphic $m-$canonical forms on $X\backslash D$ which locally at the points of D belong to $L^{2/m}$.

If $D = 0$ is the zero divisor, we write $\mathcal{L}^m_X$ and $\mathcal{L}^{2,m}_X$ instead of $\mathcal{L}^m_{X,D}$, resp. $\mathcal{L}^{2,m}_{X,D}$.

1.2. Let $\pi : Y \to X$ be a morphism of complex manifolds of the same dimension such that $\pi^{-1}(D)_{\mathrm{red}}$ is contained in an SNC divisor E on Y. Pulling back differential forms induces natural homomorphisms

$$\pi^*_m : \mathcal{L}^m_{X,D} \to \mathcal{L}^m_{Y,E} \qquad \text{and} \qquad \pi^*_{2,m} : \mathcal{L}^{2,m}_{X,D} \to \mathcal{L}^{2,m}_{Y,E}$$

(see [9, §11.1.c] for the case of π^*_m; the case of $\pi^*_{2,m}$ is similarly). They are injective, if π is dominant [9, Prop. 11.2].

For later purposes we need the following simple observation.

Lemma 1.3. *Let $\pi : Y \to X$ be a proper surjective morphism of complex manifolds of the same dimension, and let $D \subseteq X$ be an SNC divisor such that $E = \pi^{-1}(D)_{\mathrm{red}}$ is also an SNC divisor. For a section $\eta \in H^0(X\backslash D, \omega_X^{\otimes m})$ the following hold:*

$$\eta \in H^0(X, \mathcal{L}^m_{X,D}) \Leftrightarrow \pi^*\eta \in H^0(Y, \mathcal{L}^m_{Y,E})$$

and

$$\eta \in H^0(X, \mathcal{L}^{2,m}_{X,D}) \Leftrightarrow \pi^*\eta \in H^0(Y, \mathcal{L}^{2,m}_{Y,E})\,.$$

Proof. The implication '⇒' was already observed before. To show '⇐', assume that $\pi^*\eta \in H^0(Y, \mathcal{L}^m_{Y,E})$. Clearly, η is a holomorphic section of $\mathcal{L}^m_{X,D}$ if it is locally holomorphic outside an analytic subset of X of codimension 2. Hence we may assume that π is finite and that there are local coordinates $x_1, \dots, x_n$ on X, resp. $y_1, \dots, y_n$ on Y such that D is given locally by $x_1 = 0$ and π is given locally by

$$\pi : (y_1, y_2, \dots, y_n) \longmapsto (y_1^k, y_2, \dots, y_n)$$

for some $k \in \mathbb{N}$. The differential form $\omega = \frac{dx_1}{x_1} \wedge dx_2 \wedge \cdots \wedge dx_n$ is then a local generator of the invertible sheaf $\mathcal{O}_X(K_X + D)$ and so $\omega^{\otimes m}$ generates $\mathcal{L}^m_{X,D}$. Moreover,

$$\pi^*(\omega^{\otimes m}) = (k\frac{dy_1}{y_1} \wedge dy_2 \wedge \cdots \wedge dy_n)^{\otimes m}$$

locally generates $\mathcal{L}^m_{Y,E}$, and $\pi^*(x_1^a \omega^{\otimes m}) = y_1^{ak}\pi^*(\omega^{\otimes m})$ is a local section of $\mathcal{L}^{2,m}_{Y,E}$ if and only if $a \geq 1$. This easily implies both statements of the lemma. □

Applying 1.3 to local sections yields the following corollary (cf. [9, Thm. 11.1], [17, Thm. 1.1]).

Corollary 1.4. *For a bimeromorphic proper morphism $\pi : Y \to X$ of complex manifolds and D, E as above we have*

$$\pi_*(\mathcal{L}^m_{Y,E}) = \mathcal{L}^m_{X,D} \qquad \text{and} \qquad \pi_*(\mathcal{L}^{2,m}_{Y,E}) = \mathcal{L}^{2,m}_{X,D}\,.$$

1.2. **Logarithmic m-canonical forms on singular varieties.** In virtue of 1.4 the definition of the sheaves $\mathcal{L}^m_{X,D}$ and $\mathcal{L}^{2,m}_{X,D}$ can be extended as follows.

Definition 1.5. Consider a normal complex space X and a closed analytic subset $D \subseteq X$. Let $\sigma : X' \to X$ be a resolution of singularities such that $D' := \sigma^{-1}(D \cup \operatorname{Sing} X)_{\text{red}}$ is an SNC divisor. We call

$$\mathcal{L}^m_{X,D} := \sigma_*(\mathcal{L}^m_{X',D'}) \quad \text{and} \quad \mathcal{L}^{2,m}_{X,D} := \sigma_*(\mathcal{L}^{2,m}_{X',D'})$$

the sheaf of *logarithmic L^m-canonical forms*, resp. *logarithmic $L^{2,m}$-canonical forms* on X.

Because of 1.4 and by standard arguments, this is independent of the choice of resolution of singularities. As before, if $D = \emptyset$ then we write in brief $\mathcal{L}^m_X$ and $\mathcal{L}^{2,m}_X$ instead of $\mathcal{L}^m_{X,D}$ and $\mathcal{L}^{2,m}_{X,D}$, respectively.

Remark 1.6. Clearly, $D \subseteq D_1$ implies that

$$\mathcal{L}^m_{X,D} \subseteq \mathcal{L}^m_{X,D_1} \qquad \text{and} \qquad \mathcal{L}^{2,m}_{X,D} \subseteq \mathcal{L}^{2,m}_{X,D_1}\,.$$

In most of our considerations D will be a Weil divisor. This is justified by the following lemma.

Lemma 1.7. *With X, D as in (1.5) let $\operatorname{div} D$ denote the (reduced) union of all divisorial components of D. Then the following hold.*

(a) $\mathcal{L}^m_{X,D} = \mathcal{L}^m_{X,\operatorname{div}D}$ *and* $\mathcal{L}^{2,m}_{X,D} = \mathcal{L}^{2,m}_{X,\operatorname{div}D}$.

(b) *There are natural inclusions*

$$\mathcal{L}^m_{X,D} \subseteq \mathcal{O}_X\left(m(K_X + \operatorname{div}D)\right) \quad \text{and} \quad \mathcal{L}^{2,m}_{X,D} \subseteq \mathcal{O}_X\left(mK_X + (m-1)\operatorname{div}D\right)$$

with equality outside the set $\operatorname{Sing} X \cup \operatorname{Sing} \operatorname{div} D$.

Proof. (a) We confine ourselves to the proof of the first equality, the proof of the second one being similarly. By 1.6 we have $\mathcal{L}^m_{X,\mathrm{div}D} \subseteq \mathcal{L}^m_{X,D}$. To show the converse inclusion, let $\pi : X' \to X$ be a resolution of singularities such that $\Delta' := \pi^{-1}(\mathrm{div}D \cup \mathrm{Sing}\, X)_{\mathrm{red}}$ and $D' := \pi^{-1}(D \cup \mathrm{Sing}\, X)_{\mathrm{red}}$ are SNC divisors on X'. Let η be a local section of $\mathcal{L}^m_{X,D}$ in a neighbourhood, say, U of a point $p \in X$. Its restriction to $U \backslash (D \cup \mathrm{Sing}\, X)$ is a holomorphic section of $\omega_X^{\otimes m}$ and so extends holomorphically to an m-form on $U \backslash (\mathrm{div}D \cup \mathrm{Sing}\, X)$. Thus $\pi^*(\eta)$ has no poles along the components of $D' \backslash \Delta'$, so is a section of $\mathcal{O}_{X'}(m(K_{X'} + \Delta'))$, whence η is a section of $\mathcal{L}^m_{X,\mathrm{div}D}$, as required.

In order to show (b), in view of (a), we may assume that D is a reduced divisor. The sheaves $\mathcal{L}^m_{X,D}$ and $\mathcal{O}_X\,(m(K_X + D))$ are then equal outside the set $B := \mathrm{Sing}\, X \cup \mathrm{Sing}\, D$. As the latter sheaf is reflexive and $\mathrm{codim} B \geq 2$ this implies that

$$\mathcal{L}^m_{X,D} \subseteq \mathcal{O}_X\,(m(K_X + \mathrm{div}D)).$$

The same argument also gives the second inclusion in (b). □

Remark 1.8. More generally one can introduce m- and $\mathcal{L}^{2,m}$-canonical forms for any effective $\mathbb{Q}$-divisor $D = \sum a_i D_i$ with $0 < a_i \leq 1$ [1]. Again, the definition is given in two steps. If X is a manifold and $D_{\mathrm{red}} := \sum D_i$ is an SNC divisor, we set

$$\mathcal{L}^m_{X,D} := \mathcal{O}_X(mK_X + \lfloor mD \rfloor) \quad \text{and} \quad \mathcal{L}^{2,m}_{X,D} := \mathcal{O}_X(mK_X + \lfloor (m-1)D \rfloor).$$

As $a_i \leq 1$, with the same arguments as above the construction is functorial under generically finite maps of manifolds of the same dimension.

In the case of a normal variety X we choose a resolution of singularities $\sigma : X' \to X$ such that $\sigma^{-1}(D \cup \mathrm{Sing}\, X)_{\mathrm{red}}$ is a simple normal crossing divisor. Let D' denote the divisor $D^{\mathrm{pr}} + \sum_i E_i$, where E_i are the exceptional divisors and D^{pr} denotes the proper transform of D. Now one can introduce as in 1.5 the sheaves

$$\mathcal{L}^m_{X,D} := \sigma_*(\mathcal{L}^m_{X',D'}) \quad \text{and} \quad \mathcal{L}^{2,m}_{X,D} := \sigma_*(\mathcal{L}^{2,m}_{X',D'}).$$

As before one can show that this definition does not depend on the choice of the resolution.

The following proposition indicates certain functorial properties of L^m- and $L^{2,m}$-canonical forms (cf. [9, Prop. 11.3]).

Proposition 1.9. *Let $\pi : Y \to X$ be a generically finite morphism of normal connected complex spaces of the same dimension. Let $D \subseteq X$ be an analytic subset, and assume that $E \subseteq Y$ is an analytic subset with* $\mathrm{div}E = \mathrm{div}\pi^{-1}(D \cup \mathrm{Sing}\, X)$. *Then the following hold.*

(a) *There are natural injections*

$$\mathcal{L}^m_{X,D} \to \pi_*(\mathcal{L}^m_{Y,E}) \qquad \text{and} \qquad \mathcal{L}^{2,m}_{X,D} \to \pi_*(\mathcal{L}^{2,m}_{Y,E})\,.$$

(b) *If moreover π is proper, then for a form $\eta \in H^0(X \backslash D,\, \mathcal{O}_X(mK_X))$ we have*

$$\eta \in H^0(X,\, \mathcal{L}^m_{X,D}) \;\Leftrightarrow\; \pi^*\eta \in H^0(Y,\, \mathcal{L}^m_{Y,E}) \quad \textit{and}$$
$$\eta \in H^0(X,\, \mathcal{L}^{2,m}_{X,D}) \;\Leftrightarrow\; \pi^*\eta \in H^0(Y,\, \mathcal{L}^{2,m}_{Y,E})\,.$$

(c) *If π is proper and birational, then*

$$\mathcal{L}^m_{X,D} = \pi_*(\mathcal{L}^m_{Y,E}) \qquad \text{and} \qquad \mathcal{L}^{2,m}_{X,D} = \pi_*(\mathcal{L}^{2,m}_{Y,E})\,.$$

[1] In [5] such a divisor is called a *subboundary.*

Proof. Because of 1.7 we may assume that $E = \pi^{-1}(D \cup \operatorname{Sing} X)$. As before we consider only the case of $\mathcal{L}^m$-forms, the case of $\mathcal{L}^{2,m}$-forms being similarly. Consider resolutions of singularities $Y' \to Y$ and $X' \to X$ that fit into a diagram

$$\begin{tikzcd} Y' \arrow[r, "\pi'"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ Y \arrow[r, "\pi"] & X \end{tikzcd}$$

and such that

$$E' := q^{-1}(E \cup \operatorname{Sing} Y)_{\mathrm{red}} \quad \text{and} \quad D' := p^{-1}(D \cup \operatorname{Sing} X)_{\mathrm{red}}$$

are SNC divisors in Y' resp. X'. As $\pi'^{-1}(D')$ is contained in E' the morphism π' induces an injection $\mathcal{L}^m_{X',D'} \to \pi'_*(\mathcal{L}^m_{Y',E'})$ (see 1.2). Applying p_* gives the desired injection in (a).

(c) is an immediate consequence of (b). To deduce (b), note first that $\pi(\operatorname{Sing} Y) \subseteq X$ is a closed analytic subset of codimension at least 2. By 1.7 the sheaves $\mathcal{L}^m_{X,D}$ and $\mathcal{L}^m_{X,D_1}$ are equal, where $D_1 := D \cup \pi(\operatorname{Sing} Y)$. Moreover $\operatorname{Sing} Y$ is contained in $E_1 := \pi^{-1}(D_1)$ and $\mathcal{L}^m_{Y,E} \subseteq \mathcal{L}^m_{Y,E_1}$. Hence it is sufficient to prove (b) for D_1 instead of D. In other words, we may assume that E contains $\operatorname{Sing} Y$.

Let now $q : Y' \to Y$ and $p : X' \to X$ be resolutions of singularities as above. By 1.3

$$p^*\eta \in H^0(X', \mathcal{L}^m_{X',D'}) \Leftrightarrow (\pi' p)^*\eta \in H^0(Y', \mathcal{L}^m_{Y',E'}).$$

As by definition $p_*(\mathcal{L}^m_{X',D'}) = \mathcal{L}^m_{X,D}$ and $q_*(\mathcal{L}^m_{Y',E'}) = \mathcal{L}^m_{Y,E}$, (b) follows. $\square$

For our purposes it is useful to introduce certain sheaves that are invariants of the singularities.

Notation 1.10. Let X, D be as in 1.5. Because of 1.7 (b) we may form the quotient sheaves

$$\Lambda^m_{X,D} := \mathcal{O}_X\left(m(K_X + \operatorname{div} D)\right) \big/ \mathcal{L}^m_{X,D}$$

and

$$\Delta^m_{X,D} := \mathcal{O}_X\left(mK_X + (m-1)\operatorname{div} D\right) \big/ \mathcal{L}^{2,m}_{X,D}.$$

Note that by 1.7 (a) $\Lambda^m_{X,D} = \Lambda^m_{X,\operatorname{div} D}$ and $\Delta^m_{X,D} = \Delta^m_{X,\operatorname{div} D}$. Moreover, by 1.7 (b) these sheaves are concentrated on $\operatorname{Sing} X \cup \operatorname{Sing} \operatorname{div} D$. As before, in the case $D = \emptyset$ we write in brief Λ^m_X and Δ^m_X instead of $\Lambda^m_{X,D}$, resp. $\Delta^m_{X,D}$.

Later on we will need the following fact.

Lemma 1.11. *For analytic subsets $D_1 \subseteq D_2$ of X the natural maps*

$$\Lambda^m_{X,D_1} \longrightarrow \Lambda^m_{X,D_2} \quad \text{and} \quad \Delta^m_{X,D_1} \longrightarrow \Delta^m_{X,D_2}$$

are injective.

Proof. We restrict to the proof of the first inclusion the other one being similarly. By 1.7 (a) we may assume that D_1 and D_2 are reduced divisors. We need to show that

$$\mathcal{L}^m_{X,D_1} = \mathcal{L}^m_{X,D_2} \cap \mathcal{O}_X(m(K_X + D_1)).$$

The inclusion '$\subseteq$' follows from 1.6 and 1.7 (b). To show the converse inclusion, let $\pi : X' \to X$ be a resolution of singularities such that $D_i' := \pi^{-1}(D_i \cup \operatorname{Sing} X)_{\mathrm{red}}$, $i = 1, 2$, are SNC divisors. If η is a section of $\mathcal{L}^m_{X,D_2} \cap \mathcal{O}_X(m(K_X + D_1))$ defined over some open subset U of X, then $\pi^*(\eta)$ is a form in $\omega^{\otimes m}_{X'}$ that has at most

logarithmic poles along the irreducible components of D'_2 and is holomorphic along the components of the proper transform of $D_2 - D_1$. Hence $\pi^*(\eta)$ is a section of $\mathcal{O}_{X'}(m(K_{X'} + D'_1))$ so that $\eta \in H^0(U, \mathcal{L}^m_{X,D_1})$, as required. □

In the next proposition we describe the behavior of L^m- and $L^{2,m}$-canonical forms and the sheaves Λ^m and Δ^m under taking Cartesian products.

Proposition 1.12. *Let X_1, X_2 be normal complex spaces, let $D_i \subseteq X_i$, $i = 1, 2$, be closed analytic subsets and let D denote the closed analytic subset $X_1 \times D_2 \cup D_1 \times X_2$ of the product $X := X_1 \times X_2$. With $p_i : X \to X_i$ being the canonical projection ($i = 1, 2$) the following hold.*

(a) The sheaves $\mathcal{L}^m$, $\mathcal{L}^{2,m}$, Λ^m and Δ^m ($m \geq 1$) are compatible with taking products, i.e. with $\mathcal{H}$ any one of these sheaves we have

$$\mathcal{H}_{X,D} \cong p_1^*(\mathcal{H}_{X_1,D_1}) \otimes p_2^*(\mathcal{H}_{X_2,D_2}).$$

(b) If D_1 and D_2 are divisors then for all $k, m \in \mathbb{Z}$

$$\mathcal{O}_X(mK_X + kD) \cong p_1^*(\mathcal{O}_{X_1}(mK_{X_1} + kD_1)) \otimes p_2^*(\mathcal{O}_{X_2}(mK_{X_2} + kD_2))\,.$$

(c) If D_1, D_2, are divisors then D is $\mathbb{Q}$-Cartier if and only if D_1 and D_2 are $\mathbb{Q}$-Cartier.

Proof. (b) is obvious on the regular part of X. The sheaves on both sides of (b) are reflexive and so are determined by their restrictions to the regular parts, whence (b) follows.

In order to deduce (a), let $\pi_i : X'_i \to X_i$ ($i = 1, 2$) be resolutions of singularities such that $D'_i := \pi_i^{-1}(D_i \cup \operatorname{Sing} X_i)$ are SNC divisors in X'_i. The product $X' := X'_1 \times X'_2$ then provides a resolution of singularities $\pi : X' \to X$ such that $D' := \pi^{-1}(D \cup \operatorname{Sing} X) = X'_1 \times D'_2 \cup D'_1 \times X'_2$ is an SNC divisor. Let $q_i : X' \to X'_i$ denote the canonical projection ($i = 1, 2$). By (b)

$$\mathcal{L}^m_{X',D'} \cong q_1^*(\mathcal{L}^m_{X'_1,D'_1}) \otimes q_2^*(\mathcal{L}^m_{X'_2,D'_2}) \quad \text{and} \quad \mathcal{L}^{2,m}_{X',D'} \cong q_1^*(\mathcal{L}^{2,m}_{X'_1,D'_1}) \otimes q_2^*(\mathcal{L}^{2,m}_{X'_2,D'_2}).$$

Applying π_* and using the Künneth formula gives (a) for the cases $\mathcal{H} = \mathcal{L}^m$ and $\mathcal{H} = \mathcal{L}^{2,m}$. Using (b) also the remaining two cases follow by taking quotients.

(c) is an immediate consequence of the fact that for $n \in \mathbb{N}$ the divisor nD is Cartier if and only if nD_1 and nD_2 are Cartier. □

For our applications to quasihomogeneous singularities it is important to study the behaviour of L^m- and $L^{2,m}$-canonical forms under finite group actions.

Proposition 1.13. *Let G be a finite group acting on a normal complex space Y, and let $\pi : Y \to X := Y/G$ be the canonical morphism onto the orbit space. Let $D \subseteq X$ be an analytic subset and assume that π is unramified in codimension one outside $E := \pi^{-1}(D)_{\mathrm{red}}$. Then the following hold.*

(a) $\mathcal{L}^m_{X,D} = \pi_*(\mathcal{L}^m_{Y,E})^G$ *and* $\mathcal{L}^{2,m}_{X,D} = \pi_*(\mathcal{L}^{2,m}_{Y,E})^G$.

(b) $\Lambda^m_{X,D} = (\pi_*\Lambda^m_{Y,E})^G$ *and* $\Delta^m_{X,D} = (\pi_*\Delta^m_{Y,E})^G$.

(c) *If D is a divisor, then $\pi^*(D)$ is $\mathbb{Q}$-Cartier if and only if D itself is $\mathbb{Q}$-Cartier.*

Proof. In order to show (a) we may assume that D, and then also E, are Weil divisors, see 1.7. With $E' := \pi^{-1}(D \cup \operatorname{Sing} X)$, 1.9 (b) implies that $\mathcal{L}^m_{X,D} = \pi_*(\mathcal{L}^m_{Y,E'})^G$ and $\mathcal{L}^{2,m}_{X,D} = \pi_*(\mathcal{L}^{2,m}_{Y,E'})^G$. As E and E' are equal in codimension 1, (a) follows from 1.7.

(a) implies in particular that

$$
\begin{aligned}
(1)\qquad \mathcal{O}_X\left(m(K_X+D)\right) &= \pi_*(\mathcal{O}_Y\left(m(K_Y+E)\right))^G \\
(2)\qquad \mathcal{O}_X\left(mK_X+(m-1)D\right) &= \pi_*(\mathcal{O}_Y\left(mK_Y+(m-1)E\right))^G .
\end{aligned}
$$

Indeed, the involved sheaves are reflexive and so it is sufficient to verify the equalities on the part where X and Y are both smooth. Now (b) follows from (a) in view of (1) and (2). The statement of (c) is well known (see e.g., [18, 6]). □

Recall the following notion.

Definition 1.14. A morphism of complex spaces $\pi : Y \to X$ is called *non-degenerate* if for every point $y \in Y$ we have $\dim_y Y = \dim_y \pi^{-1}(\pi(y)) + \dim_{\pi(y)} X$.

For instance, every finite surjective morphism is non-degenerate. The following proposition will be used in the sequel.

Proposition 1.15. *Let $\pi : (Y,y) \to (X,x)$ be a non-degenerate morphism of normal complex space germs, and let $D \subseteq X$ be a reduced Weil divisor with preimage $E := \pi^{-1}(D)_{\text{red}}$. Then there are (in general non-canonical) injections of $\mathcal{O}_{X,x}$-modules*

$$(3)\qquad (\Lambda^m_{X,D})_x \hookrightarrow (\Lambda^m_{Y,E})_y$$

and

$$(4)\qquad (\Delta^m_{X,D})_x \hookrightarrow (\Delta^m_{Y,E})_y \,.$$

Proof. We restrict ourselves to the proof (4), the other one being similarly. First we treat the special case that π is finite. Pulling back differential forms induces an injective map

$$(5)\qquad \pi^* : \mathcal{O}_X(mK_X+(m-1)D) \hookrightarrow \mathcal{O}_Y(mK_Y+(m-1)E).$$

The analytic sets E and $\pi^{-1}(D \cup \operatorname{Sing} X)_{\text{red}}$ are equal in codimension 1 and so by 1.9 (b) for a local section, say, η of $H^0(U, \mathcal{O}_X(mK_X+(m-1)D))$ over some open subset $U \subseteq X$

$$\eta \in H^0(U, \mathcal{L}^{2,m}_{X,D}) \quad \text{if and only if} \quad \pi^*(\eta) \in H^0(\pi^{-1}(U), \mathcal{L}^{2,m}_{Y,E}).$$

Thus (5) induces an injective map as in (4).

In the general case we can find functions $f_1, \dots, f_d \in \mathcal{O}_{Y,y}$ vanishing at y, where $d := \dim_y \pi^{-1}(x)$, such that $f := (f_1, \dots, f_d)$ restricts to a finite map of germs $(\pi^{-1}(x), y) \to (\mathbb{C}^d, 0)$. Thus, letting $Z := X \times \mathbb{C}^d$ and $z := (\pi(y), 0)$, we can factorize π into two maps

$$\pi : (Y,y) \xrightarrow{\pi\times f} (Z,z) \xrightarrow{pr_1} (X,x),$$

where $\pi \times f$ is finite. With $D_Z := D \times \mathbb{C}^d$ we have (see 1.12 (a))

$$(6)\qquad (\Delta^m_{Z,D_Z})_z \cong (\Delta^m_{X,D})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z},$$

Applying the first part of the proof, (4) follows. □

1.3. **Logarithmic m-canonical forms versus log canonical, log terminal and rational singularities.** Recall the following notions.

Definition 1.16. Let X be a normal complex space and let D be a reduced effective Weil divisor on X such that the divisor $K_X + D$ is $\mathbb{Q}$-Cartier. Let $\sigma : X' \to X$ be a resolution of singularities such that $D' = \sigma^{-1}(D \cup \mathrm{Sing}\, X)_{\mathrm{red}}$ is an SNC divisor. Write

$$\sigma^*(K_X + D) = (K_{X'} + D') - \sum_i a_i E_i \qquad \text{with} \quad a_i \in \mathbb{Q} \quad \forall i,$$

where the summation is taken over the set of all divisorial irreducible components E_i of the exceptional set of the blowup σ. (The number a_i is the so called *log discrepancy* of E_i [5, 2.5.3].) One says[2] that the pair $(X,\, D)$ has

1. *log canonical singularities* if $a_i \geq 0 \quad \forall i$.
2. *log terminal singularities*[3] if $a_i > 0 \quad \forall i$ for *every* resolution $\sigma : X' \to X$.

Following [13] we will simply say that $(X,\, D)$ is log canonical (lc, for short) resp., log terminal (lt, for short). In the case when D is the zero divisor one says that X (instead of $(X, 0)$) has log canonical resp. log terminal singularities.

3. (Artin [1]) X is said to have *rational singularities* if for a resolution of singularities $\sigma : X' \to X$ the higher direct image sheaves $R^i\sigma_*(\mathcal{O}_X)$, $i \geq 1$, vanish.

In the next proposition we recall Kempf's characterization of rational singularities in terms of canonical forms. Moreover, we characterize log canonical and log terminal singularities in terms of logarithmic m-canonical forms.

Proposition 1.17. (a) (Kempf [11, Prop. on p. 50]) *X has rational singularities if and only if X is Cohen-Macaulay and $\mathcal{L}_X^{2,1} \cong \mathcal{O}_X(K_X)$ i.e., $\Delta_X^1 = 0$.*

Assume further that $K_X + D$ is a $\mathbb{Q}$-Cartier divisor. Then

(b) *$(X,\, D)$ is lc if and only if $\mathcal{L}_{X,D}^m = \mathcal{O}_X(m(K_X + D))$ for all $m \geq 1$ or, equivalently, $\Lambda_{X,D}^m = 0$.*

(c) *X is lt if and only if $\mathcal{L}_{X,D}^{2,m} = \mathcal{O}_X(mK_X + (m-1)D)$ for all $m \geq 1$ or, equivalently, $\Delta_{X,D}^m = 0$.*

Proof. (a) The original result of Kempf (formulated only for algebraic singularities) generalizes to the complex analytic setting in view of the fact that the Grauert-Riemenschneider vanishing theorem [7] also holds for non-algebraic singularities. The latter follows from a more general result of Moriwaki [16, Thm. 3.2].
(b) By assumption $k(K_X + D)$ is a Cartier divisor for some $k \in \mathbb{N}$. Fix a resolution of singularities $\sigma : X' \to X$ such that $D' := \sigma^{-1}(D \cup \mathrm{Sing}\, X)_{\mathrm{red}}$ is an SNC divisor. By 1.16, $(X,\, D)$ is lc if and only if

$$\mathcal{O}_{X'}\left(\sigma^*(k(K_X + D))\right) \subseteq \mathcal{O}_{X'}\left(k(K_{X'} + D')\right). \tag{7}$$

Thus for a local section ω of $\mathcal{O}_X(m(K_X + D))$ the tensor power $\omega^{\otimes k}$ pulls back to a section $\sigma^*\omega^{\otimes k}$ of $\mathcal{O}_{X'}(km(K_{X'} + D'))$, whence $\sigma^*(\omega)$ is a section of $\mathcal{O}_{X'}(m(K_{X'} + D'))$ and so by definition (see 1.5)

$$\mathcal{O}_X(m(K_X + D)) \subseteq \mathcal{L}_{X,D}^m \quad \text{for all } m \geq 1\,.$$

[2] See e.g. [14, Def. 2.34], [13, Def. 3.5].
[3] In [14] these singularities are called *purely log terminal.*

Since $\mathcal{L}^m_{X,D} \subseteq \mathcal{O}_X(m(K_X + D))$ (see 1.7 (b)), the latter means that

$$\Lambda^m_{X,D} = 0 \qquad \text{for all } m \geq 1 \,. \tag{8}$$

Conversely, if $\Lambda^k_{X,D} = 0$ then (7) holds, proving (b). The proof of (c) is similarly and left to the reader. □

For further purposes, it is convenient to introduce the following definition.

Definition 1.18. We will say that the pair (X, D) is *L^2-log terminal* (L^2-lt, for short) if $\Delta^m_{X,D} = 0 \quad \forall m \geq 1$.

Proposition 1.19. *Let X be a normal complex space and D be a divisor on X such that* $\operatorname{Sing} X \subseteq D$. *Assume that both K_X and D are $\mathbb{Q}$-Cartier. If (X, D) is lc, then it is L^2-lt.*

Proof. Let as above $\sigma : X' \to X$ be a resolution of singularities such that $D' := \sigma^{-1}(D)_{\mathrm{red}} = \sigma^{-1}(D \cup \operatorname{Sing} X)_{\mathrm{red}}$ is an SNC divisor. We need to show that $\mathcal{O}_X\,(mK_X + (m-1)D) \subseteq \mathcal{L}^{2;m}_{X,D}$ for all $m \in \mathbb{N}$ or, equivalently, that for every local section ω of $\mathcal{O}_X\,(mK_X + (m-1)D) \subseteq \mathcal{O}_X\,(m(K_X + D))$ the form $\sigma^*(\omega)$ extends to a section in $\mathcal{L}^m_{X',D'}(-D')$. By assumption, (8) holds and therefore $\sigma^*(\omega)$ gives a section in $\mathcal{L}^m_{X',D'}$. Choose k such that kD is a Cartier divisor, so that kD is locally given on X by one equation, say $f = 0$. The form $\omega^{\otimes k}$ is a section in $\mathcal{O}_X\,(kmK_X + k(m-1)D) = \mathcal{L}^{km}_{X,D}(-kD)$, and so it can be written locally as $f \cdot \eta$ for some section η in $\mathcal{L}^{km}_{X,D}$. Hence $\sigma^*(\omega^{\otimes k}) = \sigma^*(f\eta) = \sigma^*(f) \cdot \sigma^*(\eta)$ becomes a section of $\mathcal{L}^{km}_{X',D'}$ that vanishes along D'. It follows that $\pi^*(\omega)$ as a section of the sheaf $\mathcal{L}^m_{X',D'}$ vanishes along D' as well, which implies the assertion. □

Remark 1.20. (1) Thus for pairs (X, D) satisfying the assumptions of 1.19 the following inclusions hold:

$$(\mathrm{lt}) \subseteq (\mathrm{lc}) \subseteq (L^2 - \mathrm{lt})\,,$$

whereas for $D = 0$ by 1.17 (c) we have:

$$(\mathrm{lt}) = (L^2 - \mathrm{lt}) \subseteq (\mathrm{lc})\,.$$

The latter equality is no longer true for pairs (X, D) if $D \neq 0$. The simplest example is given by the union D of two smooth curves on a smooth surface X meeting transversally. The pair (X, D) is not lt although it is lc and hence (by 1.19) it is L^2-lt.

(2) Using the definition of log canonical forms given in 1.8 one can also characterize log canonical singularities if $D = \sum a_i D_i$ is a $\mathbb{Q}$-divisor with $0 < a_i \leq 1$. Moreover, using this characterization one can extend the notion of log canonical singularities without requiring that $K_X + D$ is $\mathbb{Q}$-Cartier.

Combining 1.9 (c) and 1.17 gives the following corollary.

Corollary 1.21. *Let $\pi : Y \to X$ be a proper surjective bimeromorphic morphism of connected normal complex spaces and let $D \subseteq X$ be an analytic subset. Denote by E the union of divisorial components of the analytic subset* $\pi^{-1}(D \cup \operatorname{Sing} X)_{\mathrm{red}}$.

(a) *If (Y, E) is lc then*

$$\mathcal{L}^m_{X,D} = \pi_*(\mathcal{O}_Y\,(m(K_Y + E)))\,.$$

(b) *If* (Y, E) *is* L^2*-lt then*

$$\mathcal{L}^{2,m}_{X,D} = \pi_*(\mathcal{O}_Y\,(mK_Y + (m-1)E))\,.$$

From 1.13 and 1.17 we obtain the following corollary.

Corollary 1.22. *Let* $\pi : Y \to X = Y/G$, $D \subseteq X$ *and* $E \subseteq Y$ *be as in 1.13.*

a) *If* (Y, E) *is* L^2*-lt then so is* (X, D).
b) *If* D *is a reduced divisor and* $\pi^*(K_X + D)$ *is* $\mathbb{Q}$*-Cartier then the same is true for the property 'lc'.*

The next corollary follows from 1.12 (a),(c) using 1.17 (b).

Corollary 1.23. *Under the assumptions as in 1.12 if* D_1, D_2 *are divisors then* $(X, D) := (X_1 \times X_2, X_1 \times D_2 \cup D_1 \times X_2)$ *is lc (resp.,* L^2*-lt) if and only if* (X_1, D_1) *and* (X_2, D_2) *are lc (resp.,* L^2*-lt).*

The following corollary is an immediate consequence of 1.15 and 1.17. Part (a) was shown by Bingener and Storch, whereas part (b) generalizes a result of Ishii and Kollár.

Corollary 1.24. *For a non-degenerate surjective morphism of normal complex spaces* $\pi : Y \to X$ *the following hold.*

(a) [2, 5.7] *If* Y *has rational singularities and* X *is Cohen-Macaulay, then* X *has also rational singularities.*
(b) (cf. [10, 1.7.II], [5, 20.3.3]) *Let* $D \subseteq X$ *be a reduced Weil divisor with preimage* $E := \pi^{-1}(D)_{\mathrm{red}}$. *If* (Y, E) *is* L^2*-lt then so is* (X, D). *If* $K_X + D$ *is* $\mathbb{Q}$*-Cartier, then the same holds for the property 'lc'.*

1.4. **Plurigenera of isolated singularities.** Let X be a normal complex space. Recall the following notions and facts [20, 12, 15].

Definition 1.25. By [20], the sections of the sheaf $\mathcal{L}^{2,m}_X$ over an open subset $U \subseteq X$ are just the sections in $H^0(U_{\mathrm{reg}}, \omega_X^{\otimes m})$ which are locally $L^{2/m}$ on U. If (X, x) is an isolated singularity then

$$\delta_m\,(X, x) = \dim_{\mathbb{C}}\,\left[\omega^{[m]}_{X,\,x}/(\mathcal{L}^{2,m}_X)_x\right] = \dim_{\mathbb{C}}\,(\Delta^m_X)_x$$

is the *m-th* L^2*-plurigenus* as defined in [20]; here $\omega^{[m]}_X := \mathcal{O}_X(mK_X)$. Similarly, the λ*-plurigenera* [15] are given by

$$\lambda_m\,(X, x) = \dim_{\mathbb{C}}\,\left[\omega^{[m]}_{X,\,x}/(\mathcal{L}^m_X)_x\right] = \dim_{\mathbb{C}}\,(\Lambda^m_X)_x\,.$$

1.26. Note that by Kempf's criterion 1.17, *an isolated singularity* (X, x) *is rational if and only if it is Cohen-Macaulay and* $\delta_1(X, x) = 0$. Moreover, *if* (X, x) *is* $\mathbb{Q}$*-Gorenstein*[4] *then by 1.17* (X, x) *is lt (resp., lc) if and only if all plurigenera* δ_m *(resp.* λ_m*),* $m \geq 1$*, vanish.*

In analogy with 1.24 we have the following result.

[4] Recall [10, 1.3-1.4] that (X, x) is Gorenstein iff it is Cohen-Macaulay and the canonical sheaf ω_X is invertible at x. It is called $\mathbb{Q}$-*Gorenstein* if the sheaf $\omega^{[m]}_X$ is invertible at x for some $m \in \mathbb{N}$.

Corollary 1.27. *If $\pi : (Y, y) \to (X, x)$ is a non-degenerate surjective morphism of germs of normal isolated singularities then*

$$\delta_m(X, x) \le \delta_m(Y, y) \quad \textit{and} \quad \lambda_m(X, x) \le \lambda_m(Y, y) \quad \forall m \ge 1\,.$$

Moreover, if $\dim_y Y > \dim_x X$ *then*

$$\delta_m(X, x) = \lambda_m(X, x) = 0 \quad \forall m \ge 1\,.$$

Proof. The first part follows from the inclusions (3) and (4) in 1.15. To show the second assertion, consider a factorization $(Y, y) \to (Z, z) \to (X, x)$ as in the proof of 1.15. Assuming that $\delta_m(X, x) \ne 0$, by (6) in the proof of *loc.cit.* the module $(\omega_Z^{[m]}/\mathcal{L}_Z^{2,m})_z = (\Delta_Z^m)_z$ has infinite dimension and is contained in the finite dimensional vector space $(\omega_Y^{[m]}/\mathcal{L}_Y^{2,m})_y = (\Delta_Y^m)_y$. Thus we get a contradiction, and so $\delta_m(X, x)$ vanishes. As $\lambda_m \le \delta_m$, the λ-plurigenera vanish as well. □

2. Logarithmic m-canonical forms on quasihomogeneous varieties

In this section we show how to compute L^m- and $L^{2,m}$-canonical forms on affine varieties with $\mathbb{C}^*$-action, and we apply this to characterize different types of singularities.

2.1. **Graded rings and associated schemes.** For the convenience of the reader we recall some facts about projective schemes associated to graded rings which will be useful in the sequel (see [4, sect. 2]).

Notation 2.1. Let K denote a field of characteristic 0 containing all roots of unity. Recall that for a finitely generated graded K-algebra $A = \bigoplus_{\nu \ge 0} A_\nu$ the associated projective scheme $\operatorname{Proj} A$ is defined by the set of all homogeneous prime ideals $\mathfrak{p}$ of A with $A_+ \not\subseteq \mathfrak{p}$, where $A_+ = \bigoplus_{\nu > 0} A_\nu$ is the augmentation ideal. The scheme $\operatorname{Proj} A$ is separated and proper over $\operatorname{Spec} A_0$ [8]. Furthermore, $\operatorname{Proj} A$ is covered by the affine open subsets

$$D_+(f) = D_+(fA) := \{\mathfrak{p} \in \operatorname{Proj} A : f \notin \mathfrak{p}\} \cong \operatorname{Spec} A_{(f)}\,,$$

where $f \in A_+$ is a homogeneous element and $A_{(f)} := (A_f)_0$ denotes the degree zero part of the localization A_f. Denote also $V = \operatorname{Spec} A$ and $V^* = V \backslash V(A_+)$ (where, as usual, $V(I)$ is the zero set of an ideal I, whereas for a homogeneous ideal $I \subseteq A$, $V_+(I)$ stands for its zero set in $\operatorname{Proj} A$). There is a natural surjective morphism $V^* \to \operatorname{Proj} A$.

The multiplicative group $\mathbb{G}_m$ of the field K acts on A via $t.a = t^\nu a$ for $t \in K^*$ and $a \in A_\nu$. If $A = A_0[A_1]$ is generated as an A_0-algebra by the elements of degree 1 then $V^* \to \operatorname{Proj} A$ is a locally trivial $\mathbb{G}_m$-bundle. In general, we have the following well known fact.

Lemma 2.2. $\operatorname{Proj} A \cong V^*/\mathbb{G}_m$.

Proof. In lack of a reference we provide the simple argument: V^* is covered by the $\mathbb{G}_m$-invariant affine open subsets $D(f) := \operatorname{Spec} A_f$, where $f \in A_d$ with $d > 0$. As the ring of invariants $(A_f)^{\mathbb{G}_m}$ is just $A_{(f)} = (A_f)_0$ we obtain $D_+(f) = D(f)/\mathbb{G}_m$ and so the lemma follows. □

2.3. To describe the situation more closely, for $f \in A_d$ with $d > 0$ denote $F = F(f) = A/(f-1)$ resp., $Y = Y(f) = \operatorname{Spec} F$ and consider the homomorphism of graded rings

$$\mu : A_f \to F[T, T^{-1}] \quad \text{given by} \quad a/f^k \longmapsto \overline{a} \cdot T^{\deg a - kd},$$

where $F[T,T^{-1}]$ is graded via $F[T,T^{-1}]_0 = F$ and $\deg T = 1$. Clearly μ is degree preserving; in particular $\mu((A_f)_0) \subseteq F$. There is a commutative diagram

$$\begin{tikzcd} A_{(f)} = (A_f)_0 \arrow[r] \arrow[d, "i"'] & F \arrow[d, "i"] \\ A_f \arrow[r, "\mu"] & F[T,T^{-1}] \end{tikzcd}$$

where i stands for the natural inclusions.

The cyclic group $\mathbb{Z}_d \cong \langle\zeta\rangle$ generated by a primitive d-th root of unity ζ acts (homogeneously) on $F[T,T^{-1}]$; namely for $b = \overline{a}T^k \in F[T,T^{-1}]$ with $a \in A$ homogeneous we let $\zeta.b = \zeta^{\deg a - k} b$. This action restricts to $F = F[T,T^{-1}]_0$; the next lemma describes the quotients.

Lemma 2.4. [4, 2.1-2.2] *(a) μ provides isomorphisms $A_f \cong F[T,T^{-1}]^{\mathbb{Z}_d}$ resp., $A_{(f)} = (A_f)_0 \cong F^{\mathbb{Z}_d}$ onto the rings of invariants, and the horizontal arrows (i.e., the orbit maps) in the induced commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} F[T,T^{-1}] = Y \times \mathbb{G}_m \arrow[r, "\mu", "/\mathbb{Z}_d"'] \arrow[d, "\mathrm{pr}_1"'] & D(f) = \operatorname{Spec} A_f \arrow[d, "/\mathbb{G}_m"] \\ \operatorname{Spec} F \quad = \quad Y \arrow[r, "/\mathbb{Z}_d"'] & D_+(f) = \operatorname{Spec} A_{(f)} \end{tikzcd}$$

are cyclic coverings; moreover, $\underline{\mu}$ is an étale covering.
(b) Furthermore, μ induces isomorphisms

$$(A_f)_{\geq 0} \cong F[T]^{\mathbb{Z}_d} \quad \textit{and} \quad (A_f)_{\leq 0} \cong F[T^{-1}]^{\mathbb{Z}_d},$$

and so it provides ramified cyclic coverings

$$Y \times \mathbb{A}^1_K \xrightarrow{/\mathbb{Z}_d} \operatorname{Spec}(A_f)_{\geq 0} \quad \textit{resp.,} \quad Y \times \mathbb{A}^1_K \xrightarrow{/\mathbb{Z}_d} \operatorname{Spec}(A_f)_{\leq 0}.$$

The following two constructions will be important in our applications below.

Example 2.5. (Weighted blowup) Let S be an indeterminate of degree -1. Consider the graded subring

$$\widehat{A} := A[S]_{\geq 0} \cong \bigoplus_{\nu \geq \mu} A_\nu S^\mu$$

of the ring $A[S]$. By definition, the *weighted blowup* of $V = \operatorname{Spec} A$ is the scheme $V' := \operatorname{Proj} \widehat{A}$. Note that for every element $fS^\mu \in \widehat{A}$ with $f \in A_\nu$ and $\nu > \mu$ we can write $(fS^\mu)^n = f \cdot (f^{n-1}S^{\mu n})$, where $\deg f^{n-1}S^{\mu n} \geq 0$ for $n \gg 0$. Therefore $D_+(fS^\mu\widehat{A}) \subseteq D_+(f\widehat{A})$, and so V' is covered by the affine open subsets $U_f := D_+(f\widehat{A}) = \operatorname{Spec} \widehat{A}_{(f)}$, where

$$\widehat{A}_{(f)} = \bigoplus_{\nu \geq 0} (A_f)_\nu S^\nu \cong (A_f)_{\geq 0} \cong F[T]^{\mathbb{Z}_d}$$

with T being an indeterminate of degree 1. Thus, $U_f \cong (Y \times \mathbb{A}^1_K)/\mathbb{Z}_d$, see 2.4 (b). In particular, if K_Y is a $\mathbb{Q}$-Cartier divisor then so is K_{U_f}, see 1.13 (c). Notice also that the blowup morphism $\sigma : V' \to V = \operatorname{Spec} A$ restricted to U_f is induced by the inclusion $A \hookrightarrow (A_f)_{\geq 0}$, and that the exceptional divisor $E = \sigma^{-1}(V(A_+))$ is isomorphic to $\operatorname{Proj} A$ under the natural morphism $V' = \operatorname{Proj} \widehat{A} \to \operatorname{Proj} A$.

Example 2.6. (Weighted completion) Let T be an indeterminate with $\deg T = 1$, and consider the projective scheme $\bar{V} = \operatorname{Proj} A[T]$. The scheme $V = \operatorname{Spec} A$ is naturally isomorphic to the affine open subset $D_+(T) \subseteq \bar{V}$ as

$$A[T]_{(T)} = \bigoplus_{\nu \geq 0} A_\nu T^{-\nu} \cong A\,.$$

Thus $\bar{V} = V \cup D_\infty$ (where $D_\infty := \{T = 0\} \cong \operatorname{Proj} A$ is the 'divisor at infinity') is indeed proper over $\operatorname{Spec} A_0$; we call $\bar{V}$ *the weighted completion* of V. The divisor D_∞ is covered by the affine open subsets $U^f := D_+(fA[T]) \simeq \operatorname{Spec}(A_f)_{\leq 0}$ with $f \in A_d,\ d > 0$ (indeed,

$$A[T]_{(f)} = (A_f[T])_0 = \bigoplus_{\nu \geq 0} (A_f)_{-\nu} T^\nu \cong (A_f)_{\leq 0}\,).$$

Furthermore, by 2.4 (b) we have $(A_f)_{\leq 0} \cong F[S]^{\mathbb{Z}_d}$, where this time $S = T^{-1}$ is an indeterminate of degree -1, and again $U^f \simeq (Y \times \mathbb{A}^1_K)/\mathbb{Z}_d$.

Remark 2.7. If, more generally, A is a graded (but not necessarily positively graded) ring, then the above construction provides a partial completion $\bar{V}$ of V.

From 2.1-2.4 we obtain the following proposition.

Proposition 2.8. *With the assumptions of 2.1, if $V^* = V \backslash V(A_+)$ is smooth then $\operatorname{Proj} A$ as well as the weighted blowup $\operatorname{Proj} A[S]_{\geq 0}$, $\deg S = -1$, have at most cyclic quotient singularities. Similarly, the subset $\bar{V} \backslash V_+(A_+A[T])$ of the weighted completion $\bar{V} = \operatorname{Proj} A[T]$, $\deg T = 1$, has at most cyclic quotient singularities.*

Proof. For a homogeneous element $f \in A_d$, $d > 0$, the morphism

$$\underline{\mu} : Y \times \mathbb{G}_m \to \operatorname{Spec} A_f = V \backslash D_f \subseteq V^*$$

is étale (see 2.4 (a)), and so $Y = \operatorname{Spec} F$ is smooth. Hence $\operatorname{Spec} A_{(f)} \cong \operatorname{Spec} \left(F^{\mathbb{Z}_d}\right) = Y/\mathbb{Z}_d$ has at most cyclic quotient singularities. As $\operatorname{Proj} A$ is covered by the affine open subsets $D_+(f) = \operatorname{Spec} A_{(f)}$, this scheme has at most cyclic quotient singularities as well. The proof of the remaining cases is similarly using the descriptions of affine open coverings given in 2.5 and 2.6. ☐

2.2. **Characterizing log canonical forms in terms of gradings.** We fix the following notation.

Notation 2.9. Let now $K = \mathbb{C}$, and let A be a normal $\mathbb{C}$-algebra of finite type with a grading $A = \bigoplus_{\nu \geq 0} A_\nu$ (which corresponds to a $\mathbb{C}^*$-action on A via $t.a = t^{\deg a} a$, where $t \in \mathbb{C}^*$ and $a \in A$ is homogeneous). In the sequel such an affine variety $V = \operatorname{Spec} A$ with $\mathbb{C}^*$-action will be referred to as a *quasihomogeneous variety* (indeed, there is a closed affine embedding $V \overset{i}{\hookrightarrow} \mathbb{C}^n$ given by a set of homogeneous generators of A and equivariant with respect to a diagonal $\mathbb{C}^*$-action on $\mathbb{C}^n$). Note that the *vertex set* $V(A_+)$ is just the fixed point set of the $\mathbb{C}^*$-action on V.

Fix a $\mathbb{C}^*$-invariant divisor D on V. In the next theorem we compute the A-modules

$$L^m_{A,D} := H^0(V, \mathcal{L}^m_{V,D}) \quad \text{and} \quad L^{2,m}_{A,D} := H^0(V, \mathcal{L}^{2,m}_{V,D})$$

as submodules of

$$H^0(V^*, \mathcal{L}^m_{V,D}) \quad \text{respectively} \quad H^0(V^*, \mathcal{L}^{2,m}_{V,D}),$$

where as before $V^* = V \backslash V(A_+)$. Note that all these modules have an induced $\mathbb{C}^*$-action and so they carry natural gradings.

Theorem 2.10. *If $V = \operatorname{Spec} A$ with $A = \bigoplus_{\nu \geq 0} A_\nu$ is a normal quasihomogeneous variety and $D \subseteq V$ is a $\mathbb{C}^*$-invariant divisor containing the divisorial part of $V(A_+)$ then*

$$L^m_{A,D} = H^0(V^*, \mathcal{L}^m_{V,D})_{\geq 0} \quad \text{and} \quad L^{2,m}_{A,D} = H^0(V^*, \mathcal{L}^{2,m}_{V,D})_{>0}\,. \tag{9}$$

2.11. Observe first that if $B := (A_f)_{\geq 0}$ with $f \in A_+$ homogeneous, then $(B_f)_{\geq 0} = B$. Bearing this in mind we start with the following special case.

Lemma 2.12. *With the notation and assumptions as in 2.10, suppose furthermore that there is an element $f \in A_d$, $d > 0$, such that $A = (A_f)_{\geq 0}$. Then $V^* = \operatorname{Spec} A_f$, and (9) holds, i.e.*

$$L^m_{A,D} = (L^m_{A_f,D^*})_{\geq 0} \qquad \text{and} \qquad L^{2,m}_{A,D} = (L^{2,m}_{A_f,D^*})_{>0}$$

with $D^* := D \cap V^*$.

Proof. As in 2.4, consider the homogeneous homomorphism

$$\mu : A = (A_f)_{\geq 0} \longrightarrow F[T] \quad \text{with} \quad a/f^k \longmapsto \bar{a} \cdot T^{\deg a - dk},$$

where T is an indeterminate with $\deg T = 1$ and $\bar{a}$ is the residue class of a in $F = A/(f-1)$. By 2.4 this homomorphism induces an étale morphism $A_f \to F[T, T^{-1}]$, and

$$A_f \cong F[T, T^{-1}]^{\mathbb{Z}_d}\,, \quad A = (A_f)_{\geq 0} \cong F[T]^{\mathbb{Z}_d}\,, \quad (A_f)_+ \cong TF[T]^{\mathbb{Z}_d}$$

with $\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}$. Geometrically this means that, with $Y = \operatorname{Spec} F$, the morphism

$$\underline{\mu} : Y \times \mathbb{C} \to V = \operatorname{Spec} A$$

gives a cyclic covering (whence $V \cong (Y \times \mathbb{C})/\mathbb{Z}_d$) non-ramified off $\underline{\mu}^{-1}(V(A_+)) = Y \times \{0\}$, so that the restriction

$$\underline{\mu}|_{Y \times \mathbb{C}^*} : Y \times \mathbb{C}^* \to V^* = V \backslash V(A_+) = \operatorname{Spec} A_f$$

is an étale covering.

To compute $L^m_{A,D}$ as a graded submodule of $L^m_{A_f,D^*}$, notice first that by 1.12 (a)

$$L^m_{F[T],\widetilde{D}} \cong L^m_{F,D\cap Y} \otimes L^m_{\mathbb{C}[T],\{0\}},$$

where

$$\widetilde{D} := \underline{\mu}^{-1}(D) = (D \cap Y) \times \mathbb{C} \cup Y \times \{0\} \subseteq Y \times \mathbb{C}$$

(note that by our assumptions $V(A_+)$ is a divisor contained in D). The module $L^m_{\mathbb{C}[T],\{0\}}$ is equal to $\mathbb{C}[T] \cdot (dT/T)^{\otimes m}$ and so it embeds into $L^m_{\mathbb{C}[T,T^{-1}]}$ as the submodule of elements of degree ≥ 0. It follows that

$$L^m_{F[T],\widetilde{D}} = (L^m_{F[T,T^{-1}],(D\cap Y)\times\mathbb{C}^*})_{\geq 0}\,.$$

Taking invariants with respect to $\mathbb{Z}_d$ and using 1.13 (a) we obtain that $L^m_{A,D} = (L^m_{A_f,D^*})_{\geq 0}$. The proof of the equality $L^{2,m}_{A,D} = (L^{2,m}_{A_f,D^*})_{>0}$ is similarly using the fact that $L^{2,m}_{\mathbb{C}[T],\{0\}}$ is generated by $T(dT/T)^{\otimes m}$. $\square$

Proof of Theorem 2.10. Let S be an indeterminate with deg $S = -1$. Consider the weighted blowup (see 2.5)

$$\sigma : V' := \mathrm{Proj}(A[S]_{\geq 0}) \to V$$

with exceptional divisor $E \subseteq V'$, and denote $D' := \sigma^{-1}(D)_{\mathrm{red}} \cup E \subseteq V'$. According to 1.9 (c) $H^0(V', \mathcal{L}^m_{V',D'}) \cong L^m_{A,D}$, and so we need to show that

$$H^0(V', \mathcal{L}^m_{V',D'}) \cong H^0(V^*, \mathcal{L}^m_{V,D})_{\geq 0}\,.$$

By 2.5 the affine open subsets

$$U_f = D_+(f\widehat{A}) = \mathrm{Spec}(A_f)_{\geq 0} \subseteq V'$$

with $f \in A_+$ homogeneous form a covering of V'. If $\omega \in H^0(V^*, \mathcal{L}^m_{V,D})$ is a homogeneous form, then $\sigma^*(\omega)$ belongs to $H^0(V', \mathcal{L}^m_{V',D'})$ if and only if, for all f as above, the form $\omega|_{D(f)} \in L^m_{A_f,D\cap D(f)}$ (where $D(f) = \mathrm{Spec}\, A_f \subset V^*$, see 2.2) extends to a form in $L^m_{(A_f)_{\geq 0},D\cap U_f}$. As by 2.11 and 2.12

$$L^m_{(A_f)_{\geq 0},D\cap U_f} = \left(L^m_{A_f,D\cap D(f)}\right)_{\geq 0}\,,$$

the result for L^m-forms follows. The proof in the case of $L^{2,m}$-forms is analogously and left to the reader. $\square$

2.13. In the next proposition we give a dual version of 2.10 which allows to control the $\mathcal{L}^m$ and $\mathcal{L}^{2,m}$-forms at infinity (this will be useful later on, see 2.25). With the assumptions as in 2.10, consider the weighted completion $\bar{V} = \mathrm{Proj}\, A[T]$ with $\deg T = 1$ (see 2.6). The subset $V_\infty := \bar{V}\backslash V_+(A_+A[T])$ contains the divisor at infinity $D_\infty \cong \mathrm{Proj}\, A$ of $\bar{V}$. Let $\bar{D}$ denote the union $D \cup D_\infty$.

Proposition 2.14. *With the above notation we have*

$$H^0(V_\infty,\, \mathcal{L}^m_{V_\infty,\bar{D}}) = H^0(V^*, \mathcal{L}^m_{V,D})_{\leq 0}$$
$$H^0(V_\infty,\, \mathcal{L}^{2,m}_{V_\infty,\bar{D}}) = H^0(V^*, \mathcal{L}^{2,m}_{V,D})_{<0}\,.$$

Proof. The variety V_∞ is covered by the affine open subsets $U^f = \mathrm{Spec}\,(A_f)_{\leq 0}$, where $f \in A_+$ is homogeneous (see 2.6). Applying 2.12 to $(A_f)_{\leq 0}$ with the grading reversed we obtain

$$L^m_{(A_f)_{\leq 0},\bar{D}} \cong (L^m_{A_f,\bar{D}})_{\leq 0} \quad \text{and} \quad L^{2,m}_{(A_f)_{\leq 0},\bar{D}} \cong (L^{2,m}_{A_f,\bar{D}})_{<0}\,.$$

Now we can proceed as in the proof of 2.10; we leave the details to the reader. $\square$

2.3. **Log terminal and log canonical singularities of quasihomogeneous varieties.** With the notations as in 2.9, let $\omega_A := H^0(V, \mathcal{O}_V(K_V))$ be the dualizing module of A and let

$$\omega_A^{[m]}(kD) := H^0(V, \mathcal{O}_V(mK_V + kD)).$$

We have seen in 1.7 (b) that $L^m_{A,D}$ and $L^{2,m}_{A,D}$ are in a natural way submodules of $\omega_A^{[m]}(mD)$ resp. $\omega_A^{[m]}((m-1)D)$. In the next theorem we identify the modules

$$\Lambda^m_{A,D} := H^0(V, \Lambda^m_{V,D}) \quad \textit{and} \quad \Delta^m_{A,D} := H^0(V, \Delta^m_{V,D})$$

with certain graded pieces of $\omega_A^{[m]}(mD)$ resp. $\omega_A^{[m]}((m-1)D)$.

Theorem 2.15. *With the notation and assumptions as in 2.10 the following hold.*

(a) *If (V^*, D^*) is lc then*

$$L^m_{A,D} \cong \omega_A^{[m]}(mD)_{\geq 0} \quad \textit{and} \quad \Lambda^m_{A,D} \cong \omega_A^{[m]}(mD)_{<0}\,.$$

(b) *Similarly, if (V^*, D^*) is L^2-lt then*

$$L^{2,m}_{A,D} \cong \omega_A^{[m]}((m-1)D)_{>0} \quad \textit{and} \quad \Delta^m_{A,D} \cong \omega_A^{[m]}((m-1)D)_{\leq 0}\,.$$

(c) *If (V^*, D^*) is lc, respectively L^2-lt then so is the weighted blowup (V', D') (see 2.5).*

Proof. (a) Observe that in virtue of 1.17 (b) $\Lambda^m_{V^*,D^*} = 0$ or, equivalently, $\mathcal{L}^m_{V^*,D^*} = \mathcal{O}_{V^*}(m(K_{V^*} + D^*))$. Hence 2.10 implies that

$$L^m_{A,D} \cong H^0(V^*, \mathcal{O}_V(m(K_V + D)))_{\geq 0}.$$

The module on the right contains $H^0(V, \mathcal{O}_V(m(K_V+D)))_{\geq 0} = \omega_A^{[m]}(mD)_{\geq 0}$, and by 1.7 (b) $\omega_A^{[m]}(mD)$ contains $L^m_{A,D}$ whence $(\omega_A^{[m]}(mD))_{\geq 0}$ contains $L^m_{A,D} = (L^m_{A,D})_{\geq 0}$. Thus $L^m_{A,D}$ and $\omega_A^{[m]}(mD)_{\geq 0}$ are equal, as required. The proof of (b) is similarly and left to the reader.

In order to show (c) consider first the special case treated in 2.12 so that $A = (A_f)_{\geq 0}$. Then $U_f = D_+(f\widehat{A}) = \operatorname{Spec}(A_f)_{\geq 0} = V$ and $\sigma|_{U_f} = \mathrm{id}$, whence $V' = V = (Y \times \mathbb{C})/\mathbb{Z}_d$ (see 2.4 (b)). If (V^*, D^*) is lc then (with the notation as in the proof of 2.12) the étale covering

$$(Y \times \mathbb{C}^*, \underline{\mu}^{-1}(D \cap V^*)) = (Y \times \mathbb{C}^*, (D \cap Y) \times \mathbb{C}^*)$$

of (V^*, D^*) is also lc. By 1.12 $(Y, D\cap Y)$ is lc and so, applying 1.12 again, $(Y \times \mathbb{C}, \widetilde{D})$ (with $\widetilde{D} = (D \cap Y) \times \mathbb{C} \cup Y \times \{0\} = \underline{\mu}^{-1}(D) \subseteq Y \times \mathbb{C})$ is lc as well. As $\underline{\mu}|_{(Y\times\mathbb{C}^*)}$ is unramified, the divisor $\underline{\mu}^*(K_V + D)$ on $Y \times \mathbb{C}$ is equal to $K_{Y\times\mathbb{C}} + \widetilde{D} + \lambda(Y \times \{0\})$ for some $\lambda \in \mathbb{Z}$; in particular, it is $\mathbb{Q}$-Cartier. Taking quotients and using 1.13 and 1.17 (b) we deduce that $(V, D) = (V', D')$ is also lc.

The general case follows easily from this with the same reasoning as in the proof of 2.10. The proof in the L^2-lt case is similarly and left to the reader. □

Corollary 2.16. *With the assumptions as in 2.10, the following hold.*

(a) *If (V^*, D^*) is lc and $K_V + D$ is $\mathbb{Q}$-Cartier then (V, D) is lc if and only if*

$$\Lambda^m_{A,D} = \omega_A^{[m]}(mD)_{<0} = 0 \quad \forall m \geq 1\,.$$

(b) *If (V^*, D^*) is L^2-lt then (V, D) is L^2-lt if and only if*

$$\Delta^m_{A,D} = \omega_A^{[m]}((m-1)D)_{\le 0} = 0 \quad \forall m \ge 1\,.$$

(c) ([4]) *If V^* has rational singularities and V is Cohen-Macaulay then V has rational singularities if and only if $(\omega_A)_{\le 0} = 0$.*

Proof. As the coherent sheaves $\Lambda^m_{V,D}$ resp., $\Delta^m_{V,D}$ on the affine variety V are globally generated, under the assumptions of (a) resp., (b) they vanish. Now (a) and (b) follow immediately from 2.15 (a), (b) and 1.17 (b)-1.18. To prove (c) observe that $\omega_A = H^0(V, \mathcal{O}_V(K_V))$. Thus (c) is a consequence of 2.15 (b) and 1.17 (a). □

2.17. Assume now that ω_A is a free A-module so that

$$\omega_A \cong A[N_A]$$

for some $N_A \in \mathbb{Z}$ (where as usual $A[N]$ is the module A equipped with the new grading $A[N]_i := A_{i+N}$). Note that by the homogeneous version of Nakayama's lemma this assumption is satisfied if, for instance, A is a Gorenstein ring and A_0 is a local ring. If moreover D is given by a homogeneous equation of degree d then

$$H^0(V, \mathcal{O}_V(m(K_V + D))) \cong A[m(N_A + d)] \qquad \text{and}$$
$$H^0(V, \mathcal{O}_V(mK_V + (m-1)D)) \cong A[mN_A + (m-1)d]\,.$$

From 2.16 we obtain the following characterizations.

Corollary 2.18. *Let (A, D) be as in 2.10. If $\omega_A \cong A[N_A]$ and D is given by a homogeneous equation of degree d then the following hold.*

(a) *If (V^*, D^*) is lc then (V, D) is lc if and only if $N_A + d \le 0$.*
(b) *If (V^*, D^*) is L^2-lt then (V, D) is L^2-lt if and only if $d > 0$ and $N_A + d \le 0$ or $d = 0$ and $N_A < 0$.*
(c) [4] *If V^* has rational singularities and A is Cohen-Macaulay then V has rational singularities if and only if $N_A < 0$.*

Example 2.19. In particular, let $V = \operatorname{Spec} A$ be a normal complete intersection of dimension n given in $\mathbb{C}^{n+s}$ by polynomials $p_1, \ldots, p_s \in \mathbb{C}^{n+s}$ which are quasihomogeneous of degrees $d_1, \ldots, d_s$ with respect to weights $w_j \ge 0,\ j = 1, \ldots, n+s$, so that

$$p_i(\lambda^{w_1} x_1, \ldots, \lambda^{w_{n+s}} x_{n+s}) = \lambda^{d_i} p(x_1, \ldots, x_{n+s}), \qquad i = 1, \ldots, s\,.$$

It is well known (see e.g. [4, p.42]) that A is Gorenstein (whence Cohen-Macaulay) and

$$\omega_A = A[N_A] \quad \text{with} \quad N_A (= N_V) := \sum_{i=1}^{s} d_i - \sum_{j=1}^{n+s} w_j\,. \tag{10}$$

Example 2.20. As a concrete example, consider the polynomial $x^a y^d + u^b + v^c = 0$ with $a, b, c, d \ge 2$. It is weighted homogeneous of degree abc with weights, say, $\deg x = bc$, $\deg y = 0$, $\deg v = ab$ and $\deg u = ac$. The associated quasihomogeneous variety $V = \{p = 0\} \subseteq \mathbb{C}^4$ has singularities only along the x- and y-axes. By 2.18 the singularities along the x-axis off the origin are log canonical iff $1/b + 1/c + 1/d \ge 1$;

they are rational and, moreover, log terminal iff $1/b+1/c+1/d>1$ (indeed, locally near a point $(x_0,0,0,0)\in V^*$ with $x_0\neq 0$ the mapping

$$V\ni(x,y,u,v)\longmapsto(x,x^{a/d}y,u,v)\in\tilde V:=\{y^d+u^b+v^c=0\}\subseteq\mathbb{C}^4$$

is well defined and biholomorphic). Letting D be the divisor $\{x=0\}$ it follows that (V,D) is lc if and only if $1/b+1/c\geq 1$. Moreover the singularities of V are rational (and log terminal) if and only if $1/a+1/b+1/c>1$ and $1/b+1/c+1/d>1$.

As another example we study varieties that are given by the maximal Pfaffians of a skew symmetric matrix (cf. [3]).

Example 2.21. Let $R=\bigoplus_{i\geq 0}R_i$ be a finitely generated graded $\mathbb{C}$-algebra which is Gorenstein with $\omega_R\cong R[N_R]$. Consider a skew symmetric $(2n+1)\times(2n+1)$-matrix (a_{ij}) of homogeneous elements of R and assume that $\deg a_{ij}=d_i+d_j-N$, where N and d_i, $1\leq i\leq 2n+1$, are positive integers satisfying $\sum_i d_i=nN$. The maximal Pfaffians generate a homogeneous ideal, say, I of R. Assume that the quotient $A:=R/I$ has dimension $\dim R-3$. Then by [3] the minimal resolution of A has the form

$$0\to R[-N]\to\bigoplus_{j=1}^{2n+1}R[-N+d_j]\xrightarrow{(a_{ij})}\bigoplus_{i=1}^{2n+1}R[-d_i]\to R\to A\to 0\,.$$

Taking $\operatorname{Ext}_R(-,\omega_R)$ of this sequence it follows that $\omega_A\cong A[N_A]$ with $N_A:=N+N_R$. Hence 2.18 applies in this situation.

2.4. **Plurigenera of quasihomogeneous singularities.** The results above provide the following explicit formulas for the plurigenera of isolated singularities of quasihomogeneous varieties, where as usual $\omega_A^{[m]}$ denotes the reflexive hull of the module $\omega_A^{\otimes m}$ of Kählerian m-differentials on A.

Proposition 2.22. *Let $A=\bigoplus_{\nu\geq 0}A_\nu$ be a normal $\mathbb{C}$-algebra of finite type and assume that the corresponding affine variety $V=\operatorname{Spec}A$ has $\dim V\geq 2$ and at most isolated singularities. Then the following hold.*

(a) *If $A_0\neq\mathbb{C}$ (in particular, if V has at least two singular points) then $\delta_m(V,p)=\lambda_m(V,p)=0$ for all $m\geq 1$ and $p\in\operatorname{Sing}V$.*
(b) *If $A_0=\mathbb{C}$ and V has a unique singular point p then for every $m\geq 1$ we have*

$$\delta_m(V,p)=\dim_{\mathbb{C}}(\omega_A^{[m]})_{\leq 0}\qquad\text{and}\qquad\lambda_m(V,p)=\dim_{\mathbb{C}}(\omega_A^{[m]})_{<0}\,.$$

Proof. (a) Let D denote the divisorial part of $V(A_+)$. Using 1.11 resp. 2.15 (b) we have an inclusion

$$\Delta_A^m\subseteq\Delta_{A,D}^m\cong(\omega_A^{[m]})((m-1)D)_{\leq 0}.$$

By our assumption, $\dim_{\mathbb{C}}A_0=\infty$. Since $(\omega_A^{[m]})((m-1)D)$ is a torsion-free A_0-module, its A_0-submodule Δ_A^m is also torsion-free over A_0, and so either $\Delta_A^m=0$ or $\dim_{\mathbb{C}}\Delta_A^m=\infty$. On the other hand,

$$\dim_{\mathbb{C}}\Delta_A^m=\sum_{p\in\operatorname{Sing}V}\delta_m(V,p)<\infty\,.$$

It follows that $\Delta_A^m=0$, and so $\delta_m(V,p)$ vanishes for all $m\geq 1$ and $p\in\operatorname{Sing}V$. Consequently $\lambda_m(V,p)=0$, and (a) follows.

(b) Since by our assumption $V_+ = \operatorname{Spec} A_0 = \{p\}$ and so $\operatorname{div} V_+ = \emptyset$, by 2.15 (a), (b) (with $D = 0$) we have

$$\Lambda_A^m \cong (\omega_A^{[m]})_{<0} \qquad \text{and} \qquad \Delta_A^m \cong (\omega_A^{[m]})_{\le 0}\,.$$

Therefore (since p is the unique singular point of V) we obtain

$$\delta_m(V,p) = \sum_{x\in \operatorname{Sing} V} \delta_m(V,x) = \dim_{\mathbb{C}}\, \Delta_A^m = \dim_{\mathbb{C}}\, (\omega_A^{[m]})_{\le 0}\,,$$

and similarly $\lambda_m(V,p) = \dim_{\mathbb{C}}\, (\omega_A^{[m]})_{<0}$, proving (b). □

Example 2.23. Let $V = \operatorname{Spec} A$ be a complete intersection as in 2.19 and assume moreover that V has an isolated singularity at the origin $0 \in \mathbb{C}^{n+s}$. As $\omega_A = A[N_A]$ we have $\omega_A^{[m]} = A[mN_A]$. Now 2.22 (a) implies the result of [15] which says that

$$\delta_m = \dim \sum_{i\le 0} A_{i+mN_A} \qquad \text{and} \qquad \lambda_m = \dim \sum_{i<0} A_{i+mN_A}\,.$$

2.5. **Log-Kodaira dimension of quasihomogeneous varieties.** Recall the following notions.

Definition 2.24. [9] Let V be a smooth quasi-projective variety, and let $\bar V$ be a smooth compactification of V by an SNC divisor $\bar D = \bar V \backslash V$. The *logarithmic plurigenera* $\bar p_m(V)$ are defined by

$$\bar p_m(V) = \dim\, H^0(\bar V,\, \mathcal{O}_{\bar V}(m(K_{\bar V} + \bar D)))\,,\;\; m \ge 1.$$

The *logarithmic Kodaira dimension* $\bar k\,(V)$ of V is

$$\bar k\,(V) = \begin{cases} -\infty & \text{if} \quad \bar p_m(V) = 0 \quad \forall m \in \mathbb{N} \\ \min\,\{k \in \mathbb{N} : \limsup_{m\to\infty} \bar p_m(V)/m^k < \infty\} & \text{otherwise.} \end{cases}$$

Taking the intersection of the modules in 2.10 (or 2.15) and 2.14 we obtain the following proposition.

Proposition 2.25. *Let $A = \bigoplus_{i\ge 0} A_i$ be a normal graded $\mathbb{C}$-algebra of finite type with $A_0 = \mathbb{C}$ (so that $V(A_+) = \{p\}$) and let $V = \operatorname{Spec} A$ be the corresponding quasihomogeneous variety. If D is a homogeneous reduced divisor on V then*

$$\bar p_m(V \backslash (D \cup \operatorname{Sing} V)) = \dim\, (L^m_{A,D})_0\,.$$

In particular, $\bar k(V \backslash (D \cup \operatorname{Sing} V)) = -\infty$ if and only if $(L^m_{A,D})_0 = 0$ for all $m \ge 1$.

Summarizing the preceding results gives the following theorem.

Theorem 2.26. *With the assumptions as in 2.25 above, suppose in addition that the divisor D is given by a homogeneous equation of degree d and that the pair (V^*, D^*) is lc (where as before, $V^* := V\backslash\{p\}$ and $D^* = D\backslash\{p\}$ with $\{p\} = V(A_+)$ being the vertex of V). Then the following hold.*

(a) $\bar p_m(V \backslash (D \cup \operatorname{Sing} V)) = \dim\, (\omega_A^{[m]})_{md}\,.$

(b) *If moreover V has an isolated singularity at p, then $\bar k(V^*) = -\infty$ if and only if $(\omega_A^{[m]})_0 = 0$ for all $m \ge 1$ or, equivalently, if and only if $\delta_m(V,\, p) = 0$ for all $m \ge 1$.*

(c) *Suppose that* K_V *is a* $\mathbb{Q}$*-Cartier divisor, so that for a certain* $m_0 \ge 1$ *we have* $\omega_A^{[m_0]} \cong A[N]$ *with some* $N \in \mathbb{Z}$. *Then*

$$\bar{k}(V \backslash (D \cup \operatorname{Sing} V)) = \begin{cases} -\infty & \text{if} \quad N + m_0 d < 0 \\ 0 & \text{if} \quad N + m_0 d = 0 \\ \dim V - 1 & \text{if} \quad N + m_0 d > 0\,. \end{cases}$$

(d) *In particular, with the assumptions as in (c)* $\bar{k}(V \backslash (D \cup \operatorname{Sing} V)) \le 0$ *if and only if* (V, D) *has a log canonical singularity at* p. *If moreover* (V, p) *is an isolated singularity then* (V, p) *is lt if and only if* $\bar{k}(V^*) = -\infty$.

Proof. By 2.15 (a) we have $L^m_{A,D} \cong \omega_A^{[m]}[md]_{\ge 0}$ and so (a) follows from 2.25. Under the assumptions as in (b), by 2.25 we have $p_m(V^*) = (\omega_A^{[m]})_0$. Thus the first equivalence in (b) is an immediate consequence of (a). To show the second equivalence, choose a non-zero homogeneous element $g \in A$ of some degree, say $k > 0$. If η is a non-zero form of degree $s < 0$ in $\omega_A^{[m]}$, then $g^s \eta^k$ is a non-vanishing form of degree 0 in $\omega_A^{[mk]}$. In other words, $(\omega_A^{[m]})_0 = 0$ for all $m \ge 1$ if and only if $(\omega_A^{[m]})_{\le 0} = 0$ for all $m \ge 1$. In view of 2.22 (a) this proves (b).

In order to deduce (c) notice that (in virtue of (a))

$$\bar{p}_m(V \backslash (D \cup \operatorname{Sing} V)) = (\omega_A^{[km_0]})_{km_0 d} = A_{k(N+m_0 d)} = 0 \quad \text{for all } k \ge 1$$

if and only if $N + m_0 d < 0$. In the case $N + m_0 d = 0$ we have $(\omega_A^{[km_0]})_{km_0 d} = A_0 = \mathbb{C}$ for all $k \ge 1$. If $N + m_0 d > 0$ then we may choose a number r such that $\dim (\omega_A^{[rkm_0]})_{rkm_0 d} = \dim A_{rk(N+m_0 d)}$ grows asymptotically like $k^{\dim A - 1} = k^{\dim \operatorname{Proj} A}$. (d) follows from 1.26 and 2.18 (a) above. ☐

Remark 2.27. Let $V = \operatorname{Spec} A$ be a normal quasihomogeneous variety with a vertex set $V_0 = \operatorname{Spec} A_0$ of positive dimension (that is, $A_0 \ne \mathbb{C}$). If V has isolated singularities then $\bar{k}(V \backslash \operatorname{Sing} V) = -\infty$ (cf. 2.22 (b)). Indeed, the general fibre of the canonical projection $q : V \to V_0$ is a smooth quasihomogeneous variety with a vertex set of dimension 0, whence it is isomorphic to $\mathbb{C}^k$ for some $k > 0$ (see e.g., [21, 8.5]).

Corollary 2.28. *Let* $A := \mathbb{C}[X_1, \ldots, X_{n+s}]/(p_1, \ldots, p_s)$ *be a normal quasihomogeneous complete intersection with weights* $w_1, \ldots, w_{n+s} > 0$ *(see 2.19). With* $V = \operatorname{Spec} A$ *being the associated affine variety and* D *being a homogeneous degree* d *divisor on* V, *we have*

$$\bar{k}(V \backslash (D \cup \operatorname{Sing} V)) = \begin{cases} -\infty & \text{if} \quad N_A + d < 0 \\ 0 & \text{if} \quad N_A + d = 0 \\ \dim V - 1 & \text{if} \quad N_A + d > 0\,. \end{cases}$$

Proof. As the weights are all positive we have $A_0 = \mathbb{C}$, moreover $\omega_A = A[N_A]$ (see (10) in 2.19), and so 2.26 (c) (with $m_0 = 1$) applies. ☐

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